Expansion of functions of one variable

Taylor's Theorem ⇒ let $f(x+h)$ is a function of h which can be expanded in ascending powers of h & is differentiable term by term any numbers of times w.r.t h, then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \infty$$

substituting $x=a$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots \infty \quad \text{—} ①$$

where $h = x-a \Rightarrow \boxed{h+a=x}$

Maclaurin's Theorem ⇒ substituting $a=0$ & $h=x$ in ①

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \infty \text{—} ②$$

then ② is known as Maclaurin's series.

Ques Expand $\sin x$ in ascending powers of $(x - \frac{\pi}{2})$

Sol^n ⇒ let $f(x) = \sin x$, $h = x - \frac{\pi}{2} \Rightarrow h = x - a \Rightarrow a = \frac{\pi}{2}$

Taylor's series is

$$f(x) = f(a) + h f'(a) + \frac{h^2 f''(a)}{2!} + \frac{h^3 f'''(a)}{3!} + \frac{h^4 f^{IV}(a)}{4!} + \dots \infty$$

$$\sin x = f(\tfrac{\pi}{2}) + (x-\tfrac{\pi}{2}) f'(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^2}{2!} f''(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} f'''(\tfrac{\pi}{2})$$
$$+ \frac{(x-\frac{\pi}{2})^4}{3!} f^{IV}(\tfrac{\pi}{2}) + \dots \infty \quad \text{—} ①$$

now ⇒ $f(x) = \sin x$; $f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x$; $f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x$; $f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x$; $f'''(\frac{\pi}{2}) = 0$

$f^{IV}(x) = \sin x$; $f^{IV}(\frac{\pi}{2}) = 1$ & so on

putting all these values in ①

$$\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} \dots \infty$$

Ques ⇒ Expand $\log(1+x)$ in powers of x. Then find series of $\log\left(\frac{1+x}{1-x}\right)$ & hence determine the value of $\log_e\left(\frac{11}{9}\right)$ up to five places of decimal.

Sol^n ⇒ $f(x) = \log(1+x)$; $f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$; $f'(0) = 1$

$f''(x) = (-1)(1+x)^{-2}$; $f''(0) = -1$

$f'''(x) = (-1)(-2)(1+x)^{-3}$; $f'''(0) = 2$

$f^{IV}(x) = (-1)(-2)(-3)(1+x)^{-4}$; $f^{IV}(0) = -6$

& so on

putting all these values in Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \dots \infty$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty \quad \text{—} ①$$

replace x by $-x$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \infty \quad \text{—} ②$$

now $\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \infty\right]$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$$

putting $x = \frac{1}{10}$, in the above result, we get

$$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{5}\left(\frac{1}{10}\right)^5 + \frac{1}{7}\left(\frac{1}{10}\right)^7 - \cdots\right]$$

$$= .20067.$$

Taylor's Theorem for a function of Two Variables

This is used to expand $f(x,y)$ in the neighbourhood of (a,b) [or expand $f(x,y)$ in powers of $(x-a)$ and $(y-b)$]

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

Maclaurin's Theorem for a function of Two Variables

This is used to expand $f(x,y)$ in the neighbourhood of origin $(0,0)$ [or expand $f(x,y)$ in powers of x and y]

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \cdots$$

Note Maclaurin's theorem is obtained by Taylor's theorem by putting $a=0$ and $b=0$.

Q.1. State the Taylor's Theorem for two variables. (2018-19)

Sol. Please see the statement above.

Q.2. Expand $e^x \log(1+y)$ in the powers of x and y upto terms of third degree. (2014-15)

Sol. $f(x,y) = e^x \log(1+y)$, $\quad f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$, $\quad f_x(0,0) = 0$

$f_y(x,y) = \frac{e^x}{1+y}$, $\quad f_y(0,0) = 1$

(Note $e^0 = 1$, $\log 1 = 0$)

$f_{xx}(x,y) = e^x \log(1+y)$, $f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \dfrac{e^x}{1+y}$, $f_{xy}(0,0) = 1$

$f_{yy}(x,y) = \dfrac{-e^x}{(1+y)^2}$, $f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x \log(1+y)$, $f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \dfrac{e^x}{1+y}$, $f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = \dfrac{-e^x}{(1+y)^2}$, $f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = \dfrac{2e^x}{(1+y)^3}$, $f_{yyy}(0,0) = 2$

Maclaurin's Theorem is given by

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{\underline{|2}}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{\underline{|3}}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

$$e^x \log(1+y) = 0 + [x.0 + y.1] + \frac{1}{2}[x^2.0 + 2xy.1 + y^2(-1)] + \frac{1}{6}[x^3.0 + 3x^2y.1 + 3xy^2(-1) + y^3.2] + \cdots$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \cdots$$

Q3. Express the function $f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$ as Taylor's Series expansion about the point $(1,2)$. (2016-17, 2017-18)

Sol. Here

$f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$, $f(1,2) = 12$

$f_x(x,y) = 2x - 9$, $f_x(1,2) = -7$

$f_y(x,y) = 6y - 9$, $f_y(1,2) = 3$

$f_{xx}(x,y) = 2$, $f_{xx}(1,2) = 2$

$f_{xy}(x,y) = 0$, $f_{xy}(1,2) = 0$

$f_{yy}(x,y) = 6$, $f_{yy}(1,2) = 6$.

Taylor's series about $(1,2)$ is given by

$$f(x,y) = f(1,2) + [(x-1) f_x(1,2) + (y-2) f_y(1,2)] + \frac{1}{\underline{|2}}[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2) f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)] + \cdots$$

$$f(x,y) = 12 - 7(x-1) + 3(y-2) + \frac{1}{2}[2(x-1)^2 + 0 + 6(y-2)^2]$$

$$= 12 - 7(x-1) + 3(y-2) + (x-1)^2 + 3(y-2)^2$$

Q4 Expand $f(x,y) = y^x$ about $(1,1)$ upto second degree terms and hence evaluate $(1.02)^{1.03}$ (2012-13)

Sol.

$f(x,y) = y^x$, $f_x(x,y) = y^x \log y$, $f_y(x,y) = x y^{x-1}$,

$f_{xx}(x,y) = y^x (\log y)^2$, $f_{yy}(x,y) = x(x-1)y^{x-2}$

$f_{xy}(x,y) = x y^{x-1} \log y + y^x . \frac{1}{y} = x y^{x-1} \log y + y^{x-1}$

at $(1,1)$;

$f(1,1) = 1$, $f_x(1,1) = 0$, $f_y(1,1) = 1$, $f_{xx}(1,1) = 0$, $f_{yy}(1,1) = 0$, $f_{xy}(1,1) = 1$

Taylor's series about $(1,1)$ upto second degree term is

$$f(x,y) = f(1,1) + [(x-1) f_x(1,1) + (y-1) f_y(1,1)] + \frac{1}{\underline{|2}}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)] + \cdots$$

$y^x = 1 + (y-1) + (x-1)(y-1) + \text{------} \quad \text{---} (1)$

Now put $y = 1.02$ and $x = 1.03$ in eq (1),

$$(1.02)^{1.03} = 1 + (0.02) + (0.03)(0.02) + \cdots$$
$$= 1 + 0.02 + 0.0006 + \text{---}$$
$$\simeq 1.0206 \quad \text{(approximately)}$$

Maxima and Minima of function of two Variables $z = f(x,y)$

Working Rule:-

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ simultaneously

Let (a,b); (c,d), — be the solution of these equations

Note- These points are called stationary points.

(iii) Find $r = \frac{\partial^2 z}{\partial x^2} = f_{xx}$; $s = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$; $t = \frac{\partial^2 z}{\partial y^2} = f_{yy}$

(iv) Find $rt - s^2$ for each stationary point, then

(a) If $rt - s^2 > 0$ and $r < 0$ for (a,b), then z has a maximum value at (a,b)

(b) If $rt - s^2 > 0$ and $r > 0$ for (a,b), then z has a minimum value at (a,b)

(c) If $rt - s^2 < 0$ for (a,b) then z has no extreme value at (a,b) and (a,b) is called a saddle point.

Note- (1) A maximum or a minimum value of a function f, is called its extreme value.

(2) If $rt - s^2 = 0$ then further investigation is required.

Q1. Find the stationary point of (2013-14)

$f(x,y) = 5x^2 + 10y^2 + 12xy - 4x - 6y + 1.$

Sol. $\frac{\partial f}{\partial x} = f_x = 10x + 12y - 4$

$\frac{\partial f}{\partial y} = f_y = 20y + 12x - 6$

For stationary point solve $f_x = 0$ and $f_y = 0$

$\Rightarrow 10x+12y-4=0$ or $5x+6y=2$ — ①

and $20y+12x-6=0$ or $6x+10y=3$ — ②

Multiplying eq ① by 6, eq ② by 5 then subtracting, we get

$30x+36y=12$

$\underline{\overset{-}{30x}+\overset{-}{50y}=\overset{-}{15}}$

$-14y=-3 \Rightarrow y=\frac{3}{14}$

Put value of y in eq ①

$5x+\frac{9}{7}=2 \Rightarrow 5x=\frac{5}{7} \Rightarrow x=\frac{1}{7}$

Stationary point is $\left(\frac{1}{7}, \frac{3}{14}\right)$.

Q2. Find the maximum value of the function $f(x,y,z)=z-2x^2-2y^2$ where $3xy-z+7=0$ (2016-17)

Sol Given $3xy-z+7=0 \Rightarrow z=3xy+7$

on putting this value of z, we get

$f(x,y)=3xy+7-2x^2-2y^2$ — ①

$f_x=3y-4x$, $f_y=3x-4y$

for stationary point, solve $f_x=0$ and $f_y=0$

$\Rightarrow 3y-4x=0$ — ② and $3x-4y=0$ — ③

on solving eq. ② and ③ we get $x=0, y=0$

$\therefore (0,0)$ is the stationary point.

Now

$r=f_{xx}=-4$, $s=f_{xy}=3$, $t=f_{yy}=-4$

$rt-s^2=(-4)(-4)-(3)^2=16-9=7>0$

$\because rt-s^2>0$ and $r<0$

$\therefore f(x,y)$ has maximum value at $(0,0)$

Put $x=0, y=0$ in eq ①, Maximum value is

Max. $f=0+7-0-0=7$ Ans.

Q3. Locate the stationary point of $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature. (2012-13)

Sol Let $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

$f_x=4x^3-4x+4y$

$f_y=4y^3+4x-4y$

Now solve $f_x=0$ and $f_y=0$

$\Rightarrow 4x^3-4x+4y=0$ or $x^3-x+y=0$ — ①

and $4y^3+4x-4y=0$ or $y^3+x-y=0$ — ②

Adding eqn. ① and ②, we get

$x^3+y^3=0 \Rightarrow x^3=-y^3 \Rightarrow x=-y$ or $y=-x$

Now put $y=-x$ in eq ①,

$x^3-x-x=0$ or $x^3-2x=0$ or $x(x^2-2)=0$

$\Rightarrow x=0, \pm\sqrt{2}$

$y=-x$ gives $(0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2})$ as stationary points.

Now

$r=f_{xx}=12x^2-4$; $s=f_{xy}=4$; $t=f_{yy}=12y^2-4$

Stationary Point	r	s	t	$rt-s^2$	Nature	Max/Min Value
$(0,0)$	-4	4	-4	$12>0$	Maximum (as $r<0$)	0
$(\sqrt{2},-\sqrt{2})$	20	4	20	$384>0$	Minimum (as $r>0$)	-8
$(-\sqrt{2},\sqrt{2})$	20	4	20	$384>0$	Minimum (as $r>0$)	-8

Hence Maxima at $(0,0)$, maximum value $=0$

Minima at $(\sqrt{2},-\sqrt{2})$, Minimum value $=-8$

Minima at $(-\sqrt{2},\sqrt{2})$, Minimum value $=-8$

Q4. Find the extreme values of function x^3+y^3-3axy. (2018–19)

Sol. Here $f(x,y)=x^3+y^3-3axy$

$f_x = 3x^2-3ay$, $f_y = 3y^2-3ax$

$r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = f_{yy} = 6y$

Now $f_x = 0$ and $f_y = 0$

$\Rightarrow x^2 - ay = 0$ —① and $y^2 - ax = 0$ —②

From ①, $y = \frac{x^2}{a}$ —③. Put this value of y in ②,

$\frac{x^4}{a^2} - ax = 0$ or $x(x^3-a^3)=0$ or $x = 0, a$

When $x=0, y=0$; when $x=a$, $y=a$ (using eq ③)

$\therefore$ There are two stationary points $(0,0)$ and (a,a).

Now $rt - s^2 = 36xy - 9a^2$

At $(0,0)$ $rt-s^2 = -9a^2 < 0$ There is no extreme value at $(0,0)$

At (a,a) $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$; $a \neq 0$

$\Rightarrow f(x,y)$ has extreme value at (a,a)

Now at (a,a), $r = 6a$

(i) If $a>0, r>0$ so that $f(x,y)$ has a minimum value at (a,a)

Minimum value $= a^3+a^3-3a^3 = -a^3$

(ii) If $a<0$, $r<0$ so that $f(x,y)$ has a maximum value at (a,a)

Maximum value $= (-a)^3+(-a)^3-3(-a)(-a)(-a)$
$= -a^3-a^3+3a^3 = a^3$.

Ans

Q5. Find the stationary point of $f(x,y)=x^3+y^3+3axy$, $(a>0)$ (2018–19)

Sol. Proceed as above

Ans $(0,0)$ and $(-a,-a)$ are stationary points.

Lagrange's Method of Undetermined Multipliers

Let $f(x,y,z)$ be a function of x,y,z which is to be examined for maximum or minimum value. Also let the variables x,y,z be connected by the relation $\phi(x,y,z)=0$.

Then construct Lagrange's function $F(x,y,z)$ as

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

Stationary point is given by $dF = 0$

$\Rightarrow \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$

$\Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$ —①

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$ —②

$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$ —③

On solving eqn. ①, ② and ③ we get values of x, y, z and λ.

Note- Lagrange's method does not enable us to find whether there is a maximum or minimum. This fact is determined by the physical consideration of the given problem.

Q1. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (2013–14)

Sol: Let x, y, z be three parts of 24, such that

$x + y + z = 24$ —— ①

We have to maximize $f(x,y,z) = z \cdot y^2 x^3$

(where z is I^{st}, y is II^{nd} and x is III^{rd} part of 24)

Now Lagrange's Function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ → Relation

$F(x,y,z) = zy^2x^3 + \lambda(x+y+z-24)$

For stationary point, $dF = 0$

$\Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$

$\Rightarrow [3zy^2x^2 + \lambda]dx + [2zyx^3 + \lambda]dy + [y^2x^3 + \lambda]dz = 0$

$\Rightarrow 3zy^2x^2 + \lambda = 0$ —— ②

$2zyx^3 + \lambda = 0$ —— ③

$y^2x^3 + \lambda = 0$ —— ④

Now multiplying eq. ② by x, ③ by y, ④ by z and adding,

$6zy^2x^3 + \lambda(x+y+z) = 0$ (using ①, $x+y+z=24$)

$\Rightarrow \boxed{\lambda = -\frac{zy^2x^3}{4}}$

eq② $\Rightarrow 3zy^2x^2 - \frac{zy^2x^3}{4} = 0 \Rightarrow zy^2x^2\left(3 - \frac{x}{4}\right) = 0 \Rightarrow 3 = \frac{x}{4}$

$\Rightarrow \boxed{x = 12}$

eq③ $\Rightarrow 2zyx^3 - \frac{zy^2x^3}{4} = 0 \Rightarrow zyx^3\left(2 - \frac{y}{4}\right) = 0 \Rightarrow 2 = \frac{y}{4} \Rightarrow \boxed{y = 8}$

eq④ $\Rightarrow y^2x^3 - \frac{zy^2x^3}{4} = 0 \Rightarrow y^2x^3\left(1 - \frac{z}{4}\right) = 0 \Rightarrow 1 = \frac{z}{4} \Rightarrow \boxed{z = 4}$

Hence required parts of 24 are 4, 8, 12.

Q. Divide a number into three parts such that the product of first, square of the second and the cube of third is maximum. (2016-17)

Hint: Let number is a, then proceed as above.

Answer is $\frac{a}{6}, \frac{a}{3}, \frac{a}{2}$

Q3. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol: Let (x,y,z) be a vertex of the parallelopiped then it lies on the given ellipsoid

$\therefore \phi(x,y,z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ —— ①

Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in ellipsoid

$\therefore$ Volume $V = 8xyz$ —— ②

Consider Lagrange's Function

$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

For stationary point $dF = 0 \Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$

$\Rightarrow \left[8yz + \lambda \cdot \frac{2x}{a^2}\right]dx + \left[8xz + \lambda\frac{2y}{b^2}\right]dy + \left[8xy + \lambda\frac{2z}{c^2}\right]dz = 0$

$\Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0$ —— ③, $8xz + \frac{2y\lambda}{b^2} = 0$ —— ④, $8xy + \frac{2z\lambda}{c^2} = 0$ —— ⑤

Multiplying ③, ④, ⑤ by x, y, z respectively and adding

$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$

$\Rightarrow 24xyz + 2\lambda = 0 \Rightarrow \lambda = -12xyz$ (using ①)

eq③ $\Rightarrow 8yz - 12xyz \cdot \frac{2x}{a^2} = 0 \Rightarrow 8yz\left(1 - \frac{3x^2}{a^2}\right) = 0 \Rightarrow$ ($\because y, z \neq 0$)

$\Rightarrow 1 - \frac{3x^2}{a^2} = 0 \Rightarrow x = \frac{a}{\sqrt{3}}$

Similarly from ④ and ⑤, we get $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$

$\therefore$ Volume of the largest rectangular parallelopiped

$= 8xyz = \frac{8abc}{3\sqrt{3}}$ Ans.

Imp

Q4. Find the maximum and minimum distance of the point $(1, 2, -1)$ from the sphere $x^2+y^2+z^2=24$. (2017-18) (2018-19)

Sol. Let (x, y, z) be any point on the sphere. Distance of the point $A(1, 2, -1)$ from (x, y, z) is given by $\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2$ — ①

Subject to the condition that $\phi(x,y,z) \equiv x^2+y^2+z^2-24=0$ — ②

Consider Lagrange's function

$$F(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)$$

For stationary point $dF=0$

$\Rightarrow [2(x-1)+2\lambda x]\,dx + [2(y-2)+2\lambda y]\,dy + [2(z+1)+2\lambda z]\,dz=0$

$\Rightarrow 2(x-1)+2\lambda x=0$ or $(x-1)+\lambda x=0$ — ③

$2(y-2)+2\lambda y=0$ or $(y-2)+\lambda y=0$ — ④

$2(z+1)+2\lambda z=0$ or $(z+1)+\lambda z=0$ — ⑤

Multiplying ③ by x, ④ by y, ⑤ by z and adding, we get

$(x^2+y^2+z^2)-x-2y+z+\lambda(x^2+y^2+z^2)=0$

or $x+2y-z = 24(1+\lambda)$ — ⑥ (using ②)

From ③, ④ and ⑤, $x=\frac{1}{1+\lambda}$, $y=\frac{2}{1+\lambda}$, $z=\frac{-1}{1+\lambda}$

Putting these values of x, y, z in ⑥, we get

$$\frac{1}{1+\lambda}+\frac{4}{1+\lambda}+\frac{1}{1+\lambda} = 24(1+\lambda)$$

or $(1+\lambda)^2=\frac{1}{4}$ or $1+\lambda=\pm\frac{1}{2}$

$\therefore 1+\lambda=\frac{1}{2} \Rightarrow x=2, y=4, z=-2$; Point $P(2,4,-2)$

$1+\lambda=-\frac{1}{2} \Rightarrow x=-2, y=-4, z=2$; Point $Q(-2,-4,2)$

Now $AP=\sqrt{(2-1)^2+(4-2)^2+(-2+1)^2}=\sqrt{1+4+1}=\sqrt{6}$

and $AQ=\sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2}=\sqrt{9+36+9}=\sqrt{54}=3\sqrt{6}$

$\therefore P(2,4,-2)$ is at a minimum distance from A and the minimum distance $=\sqrt{6}$

$Q(-2,-4,2)$ is at a maximum distance from A and the maximum distance $=3\sqrt{6}$.

Q5. Find the maximum and minimum distances from the origin to the curve $x^2+4xy+6y^2=140$. (2011-12)

Sol. Let (x,y) be any point on the curve. Distance of the point $A(0,0)$ from (x,y) is given by $\sqrt{x^2+y^2}$

If the distance is maximum or minimum, so will be the square of the distance.

Let $f(x,y) = x^2+y^2$ — ①

Subject to the condition

$\phi(x,y) \equiv x^2+4xy+6y^2-140=0$ — ②

Consider Lagrange's function

$$F(x,y) = f(x,y)+\lambda\phi(x,y)$$
$$= x^2+y^2+\lambda(x^2+4xy+6y^2-140)$$

For stationary values $dF=0 \Rightarrow \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=0$

$\Rightarrow [2x+\lambda(2x+4y)]\,dx+[2y+\lambda(4x+12y)]\,dy=0$

$\Rightarrow 2x+\lambda(2x+4y)=0$ or $x+\lambda(x+2y)=0$ — ③

and $2y+\lambda(4x+12y)=0$ or $y+\lambda(2x+6y)=0$ — ④

Multiplying eq ③ by x, ④ by y and adding

$x^2+y^2+\lambda(x^2+4xy+6y^2)=0$

$\Rightarrow f+\lambda(140)=0 \Rightarrow \lambda=-\frac{f}{140}$, using ① & ②

eq ③ $\Rightarrow x - \frac{f}{140}(x+2y) = 0 \Rightarrow (140-f)x - 2fy = 0$ — ⑤

eq ④ $\rightarrow y - \frac{f}{140}(2x+6y) = 0 \Rightarrow -fx + (70-3f)y = 0$ — ⑥

Solving ⑤ and ⑥, we get

$f^2 - 490f + 9800 = 0$

$\Rightarrow f = \frac{490 \pm \sqrt{(490)^2 - 4(9800)}}{2} = 245 \pm 35\sqrt{41}$

$= 469.1093 \ , \ 20.8906$

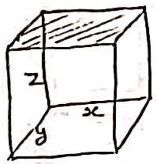
Hence maximum distance $= \sqrt{469.1093} = 21.6589$

and minimum distance $= \sqrt{20.8906} = 4.5706$

Imp

Q. A rectangular box, open at the top is to have capacity of 32 c.c. Determine, using Lagrange's method of multipliers, the dimensions of the box such that the least material is required for the construction of the box. (2014-15)

Sol Let x, y, z be the dimensions of the rectangular box, open at the top.



Volume $V = xyz = 32$ (given)

Surface area of the box is

$S = xy + 2yz + 2zx$ (which is to be minimized)

Consider Lagrange's function

$F(x,y,z) = xy + 2yz + 2zx + \lambda(xyz - 32)$

for stationary point $dF = 0$

$\Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$

$\Rightarrow (y + 2z + \lambda yz)dx + (x + 2z + \lambda xz)dy + (2y + 2x + \lambda xy)dz = 0$

$\Rightarrow y + 2z + \lambda yz = 0$ — ①

$x + 2z + \lambda xz = 0$ — ②

$2y + 2x + \lambda xy = 0$ — ③

Multiplying eq. ①, ②, ③ by x, y, z respectively, we get

$xy + 2xz + \lambda xyz = 0$ — ④

$xy + 2yz + \lambda xyz = 0$ — ⑤

$2yz + 2zx + \lambda xyz = 0$ — ⑥

eq ④ − eq ⑤ $\Rightarrow 2xz - 2yz = 0 \Rightarrow 2z(x-y) = 0 \Rightarrow x = y$

eq ⑤ − eq ⑥ $\Rightarrow xy - 2zx = 0 \Rightarrow x(y - 2z) = 0 \Rightarrow y = 2z$

Hence $x = y = 2z$

i.e. Length = breadth = 2 height

Now given $xyz = 32$

$\Rightarrow (2z)(2z)z = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2$

Hence $x = y = 4 \ , \ z = 2$ Ans.

V Imp

Q7. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when (2015-16)

(i) box is open at the top

(ii) box is closed

Sol Let x, y, z be the length, breadth and height of rectangular box respectively, then

Volume $V = xyz$ (which is to be maximised)

(i) When box is open at the top

Surface area $S = xy + 2yz + 2zx$

(ii) When box is closed

Surface area $S = 2xy + 2yz + 2zx$

Taking surface area $S = nxy + 2yz + 2zx$ (S is given)
here $n=1$ if box is open at top and $n=2$ if box is closed

Consider Lagrange's function
$$F(x,y,z) = xyz + \lambda(nxy + 2yz + 2zx - S)$$

For stationary point $dF = 0$

$\Rightarrow [yz + \lambda(ny+2z)]dx + [xz + \lambda(nx+2z)]dy + [xy + \lambda(2y+2x)]dz = 0$

$\Rightarrow yz + \lambda(ny + 2z) = 0$ —— ①

$xz + \lambda(nx + 2z) = 0$ —— ②

$xy + \lambda(2y + 2x) = 0$ —— ③

Multiplying eq. ①, ②, ③ by x, y, z respectively, we get

$xyz + \lambda(nxy + 2xz) = 0$ —— ④

$xyz + \lambda(nxy + 2yz) = 0$ —— ⑤

$xyz + \lambda(2yz + 2xz) = 0$ —— ⑥

eq ④ − eq ⑤ $\Rightarrow \lambda(2xz - 2yz) = 0 \Rightarrow 2z\lambda(x-y) = 0 \Rightarrow x = y$

eq ⑤ − eq ⑥ $\Rightarrow \lambda(nxy - 2xz) = 0 \Rightarrow x\lambda(ny - 2z) = 0 \Rightarrow ny = 2z$

(i) When box is open at the top, put $n=1$

$\therefore x = y$ and $y = 2z$ $\Rightarrow x = y = 2z$

Hence $S = x^2 + x^2 + x^2 = 3x^2$

$\therefore$ dimensions are $x = \sqrt{\frac{S}{3}} = y$, $z = \frac{1}{2}x = \frac{1}{2}\sqrt{\frac{S}{3}}$

(ii) When box is closed, put $n=2$

$\therefore x = y$ and $2y = 2z \Rightarrow y = z$, $\Rightarrow x = y = z$

Hence $S = 2x^2 + 2x^2 + 2x^2 = 6x^2$

$\therefore$ dimensions are $x = \sqrt{\frac{S}{6}} = y = z$

Ans

Jacobians

Definitions-

(i) If u and v are the functions of two variables x and y then Jacobian (or functional determinant) of u and v with respect to x and y is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(u,v)$ and given by

$$\frac{\partial(u,v)}{\partial(x,y)} = J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

(ii) If u, v and w are the functions of three independent variables x, y and z then the Jacobian of u, v, w with respect to x, y and z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J(u,v,w) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Note: Similarly if $u_1, u_2, \ldots, u_n$ are the functions of $x_1, x_2, \ldots, x_n$ then we can find $J(u_1, u_2, \ldots, u_n)$

Q1 If $u = x(1-y)$, $v = xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$ (2019-20)

Sol: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$

$= x(1-y) + xy = x$

Q2. If $x = e^{v}\sec u$, $y = e^{v}\tan u$ then evaluate $\frac{\partial(x,y)}{\partial(u,v)}$ (2020-21)

Sol

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} e^{v}\sec u\tan u & e^{v}\sec u \\ e^{v}\sec^2 u & e^{v}\tan u \end{vmatrix}$$

$$= e^{2v}\sec u\tan^2 u - e^{2v}\sec^3 u$$

$$= e^{2v}\sec u(\tan^2 u - \sec^2 u) \qquad (\because 1+\tan^2 u = \sec^2 u)$$

$$= e^{2v}\sec u(-1) = -e^{2v}\sec u$$

Imp

Q3. If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_1x_3}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$, then find the value of $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$ (2017-18)

Sol

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix}$$

$$= \frac{(x_2x_3)(x_1x_3)(x_1x_2)}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad \left(\because \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}\right)$$

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = (1)(-1)(0-4) = 4$$

Q4. If $x+y+z = u$, $y+z = uv$, $z = uvw$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ (2015-16)

Sol,

Put $z = uvw$ in $y+z = uv$; $y = uv - uvw = uv(1-w)$

Put $y+z = uv$ in $x+y+z = u$; $x = u - uv = u(1-v)$

$\therefore x = u(1-v)$, $y = uv(1-w)$ and $z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} \qquad \text{Applying } R_1 \to R_1 + (R_2+R_3)$$

$$= u^2v(1-w) + u^2vw = u^2v.$$

Properties of Jacobians (Chain Rules)

(1) If u, v are functions of r, s where r, s are functions of x, y then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}}$$

(2) If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v

then $J_1 J_2 = 1$, i.e., $\boxed{\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1}$

Q1 If $x = u(1+v)$, $y = v(1+u)$ then find the Jacobian of u, v with respect to x, y. (2017-18)

Sol

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv = 1+u+v$$

By chain rule $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{1+u+v}$

Q2. If $x = r\cos\theta$, $y = r\sin\theta$, $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ then show that $JJ^* = 1$

Or

Verify the chain rule for Jacobians if $x = r\cos\theta$, $y = r\sin\theta$.

Sol $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r$$

Now $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

and $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

Hence $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1 \Rightarrow JJ^* = 1$

Thus chain rule is verified.

Q3. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ if $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$. (2014-15)

Sol.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\sqrt{\frac{w}{v}}\sqrt{\frac{u}{w}}\sqrt{\frac{v}{u}} + \frac{1}{8}\sqrt{\frac{v}{w}}\sqrt{\frac{w}{u}}\sqrt{\frac{u}{v}} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} u_r & u_\theta & u_\phi \\ v_r & v_\theta & v_\phi \\ w_r & w_\theta & w_\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \left\{ \cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} \right\}$$

$$= r^2\sin\theta \{ \cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi) \}$$

$$= r^2\sin\theta \{ \cos^2\theta + \sin^2\theta \} = r^2\sin\theta$$

Therefore, by chain rule of Jacobians

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \frac{1}{4} r^2\sin\theta .$$

Q4. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ then find $\frac{\partial(r,\theta,z)}{\partial(x,y,z)}$ (2018-19)

Sol.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

By chain rule

$$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(r,\theta,z)}} = \frac{1}{r}$$

Q5. Calculate $\frac{\partial(u,v)}{\partial(x,y)}$ for $x = e^u\cos v$ and $y = e^u\sin v$ (2011-12)

Sol. Proceed as above.

($Ans = e^{-2u}$)

Jacobian of Implicit Functions

If u_1, u_2, u_3 are the implicit functions of x_1, x_2, x_3 (i.e, we can't express u_1, u_2, u_3 in terms of x_1, x_2, x_3) then take:

$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

then

$$\boxed{\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3\left[\frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \bigg/ \frac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)}\right]}$$

Note- If u_1, u_2 are the implicit functions of x_1, x_2 then $F_1(u_1, u_2, x_1, x_2) = 0$ and $F_2(u_1, u_2, x_1, x_2) = 0$

$$\boxed{\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2\left[\frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} \bigg/ \frac{\partial(F_1, F_2)}{\partial(u_1, u_2)}\right]}$$

Q1. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ then show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u-v)}$$

Sol. Here u, v are implicit function.

$F_1(u, v, x, y) = u^3 + v^3 - x - y = 0$

$F_2(u, v, x, y) = u^2 + v^2 - x^3 - y^3 = 0$

$$\frac{\partial(F_1, F_2)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} = 3y^2 - 3x^2$$

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6u^2v - 6uv^2$$
$$= 6uv(u-v)$$

Hence

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2\left[\frac{\partial(F_1, F_2)}{\partial(x, y)} \bigg/ \frac{\partial(F_1, F_2)}{\partial(u, v)}\right]$$
$$= \frac{3(y^2 - x^2)}{6uv(u-v)} = \frac{y^2 - x^2}{2uv(u-v)}$$

V Imp

Q2. If u, v, w are the roots of the equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ then find $\frac{\partial(u, v, w)}{\partial(a, b, c)}$ (2018-19)

Sol. If u, v, w are the roots of the equation

$(x-a)^2 + (x-b)^3 + (x-c)^3 = 0$

$(x^3 - a^3 + 3xa^2 - 3x^2a) + (x^3 - b^3 + 3xb^2 - 3x^2b) + (x^3 - c^3 + 3xc^2 - 3x^2c) = 0$

$\Rightarrow 3x^3 - 3x^2(a+b+c) + 3x(a^2+b^2+c^2) - (a^3+b^3+c^3) = 0$

Then $u + v + w = \frac{3(a+b+c)}{3}$

$uv + vw + wu = \frac{3(a^2+b^2+c^2)}{3}$

$uvw = \frac{a^3+b^3+c^3}{3}$

Let $F_1 \equiv u+v+w-a-b-c=0$

$F_2 \equiv uv+vw+wu-a^2-b^2-c^2=0$

$F_3 \equiv uvw-\frac{1}{3}(a^3+b^3+c^3)=0$

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3\left[\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}\right] \quad \text{—(1)}$$

Now

$$\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -2a & -2(b-a) & -2(c-a) \\ -a^2 & -(b^2-a^2) & -(c^2-a^2) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$= (-1)\{-2(b-a)\}\{-2(c-a)\}\begin{vmatrix} 1 & 0 & 0 \\ 2a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$= -2(b-a)(c-a)(c+a-b-a)$

$= -2(b-a)(c-a)(c-b)$

$= -2(a-b)(b-c)(c-a)$

Also

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$= (u-v)(u-w)\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

$= (u-v)(u-w)(v-w) = -(u-v)(v-w)(w-u)$

Hence from eq (1),

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = -\left[\frac{2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}\right]$$

Q3. If u, v, w are the roots of the equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ in λ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ (2015-16)

Sol. Proceed as in last question.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Q4. If $u^3+v^3+w^3=x+y+z$, $u^2+v^2+w^2=x^3+y^3+z^3$ and $u+v+w=x^2+y^2+z^2$, then show that (2019-20)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Hint: Proceed as above

Q5. If $u^3+v+w=x+y^2+z^2$
$u+v^2+w=x^2+y+z^2$
$u+v+w^3=x^2+y^2+z$

Show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4xy(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$$ (2020-21)

HINT: Proceed as above

Functional Relationship-

① If $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} \neq 0$ then functions u, v, w are independent of variables x, y, z and no relation between u, v, w exists.

② If $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ then the functions u, v, w of three independent variables x, y, z are not independent (i.e. dependent) and a relationship exists between u, v and w.

Q(1) Are the functions $u = \dfrac{x-y}{x+z}$, $v = \dfrac{x+z}{y+z}$ functionally dependent? If so, find the relation between them. (2011-12)

Sol. Note that here u, v are taking functions of any two variables (say x and z), therefore z is treating as a constant

Now $\dfrac{\partial u}{\partial x} = \dfrac{(x+z)-(x-y)}{(x+z)^2} = \dfrac{y+z}{(x+z)^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{1}{x+z}(0-1) = -\dfrac{1}{x+z}$

$\dfrac{\partial v}{\partial x} = \dfrac{1}{y+z}(1+0) = \dfrac{1}{y+z}$

$\dfrac{\partial v}{\partial y} = \dfrac{-1}{(y+z)^2}(x+z) = \dfrac{-(x+z)}{(y+z)^2}$

Now $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x}$

$= \dfrac{-(y+z)}{(x+z)^2}\,\dfrac{(x+z)}{(y+z)^2} + \dfrac{1}{(x+z)}\cdot\dfrac{1}{(y+z)}$

$= -\dfrac{1}{(x+z)(y+z)} + \dfrac{1}{(x+z)(y+z)} = 0$

Hence, the functional relationship exists between u and v.

Now $1-u = 1 - \dfrac{x-y}{x+z} = \dfrac{x+z-x+y}{x+z} = \dfrac{y+z}{x+z} = \dfrac{1}{v}$

Hence. $\boxed{1-u = \frac{1}{v} \text{ or } \frac{1}{1-u} = v}$

Imp

Q2. If $u = x+2y+z$, $v = x-2y+3z$ and $w = 2xy - xz + 4yz - 2z^2$. Show that they are not independent. Find the relation between u, v and w. (2016-17)

Sol.
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix}$$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & 2x-4y+6z & -x+2y-3z \end{vmatrix}$$

$= -4(-x+2y-3z) - 2(2x-4y+6z)$

$= 4x - 8y + 12z - 4x + 8y - 12z$,

$= 0$

Hence u, v, w are not independent.

Now $u+v = 2x+4z$
$u-v = 4y-2z$

Multiplying these, we get

$(u+v)(u-v) = (2x+4z)(4y-2z)$
$= 4(2xy+4yz-zx-2z^2)$
$\Rightarrow \boxed{u^2-v^2 = 4w}$

Q3. Show that $u = y+z$, $v = x+2z^2$, $w = x-4yz-2y^2$ are not independent. Find the relation between them. (2013-14)

Sol. Relation is $2u^2 = v-w$

Hint: $J(u,v,w)=0$ shows u,v,w are not independent.

Q4. Show that the functions

$u = x+y+z$
$v = x^2+y^2+z^2-2xy-2yz-2zx$
$w = x^3+y^3+z^3-3xyz$

are functionally related. Find the relation between them. (2013-14, 2017-18)

Sol $J(u,v,w)=0$ shows u,v,w are dependent and relation is $w = \dfrac{u(u^2+3v)}{4}$

Proceed as above.

Approximation of Errors

If δx and δy are small change (or errors) in x and y respectively, then an approximate change (or error) in z is δz.

Note- ① If δx is the error in x, then

② relative error $= \dfrac{\delta x}{x}$

③ percentage error $= \dfrac{\delta x}{x} \times 100$.

Q.1. The period of a simple pendulum is $T = 2\pi\sqrt{\dfrac{l}{g}}$. Find the maximum(%) error in T due to the possible errors upto 1% in l and 2.5% in g. (2013-14)

Sol $T = 2\pi\sqrt{\dfrac{l}{g}}$

Taking log, $\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$

Differentiating, we get

$\dfrac{1}{T}\delta T = \dfrac{1}{2}\dfrac{\delta l}{l} - \dfrac{1}{2}\dfrac{\delta g}{g}$

$\Rightarrow \dfrac{\delta T}{T}\times 100 = \dfrac{1}{2}\left[\dfrac{\delta l}{l}\times 100 - \dfrac{\delta g}{g}\times 100\right]$
$= \dfrac{1}{2}(1 \pm 2.5)$

Given
$\dfrac{\delta l}{l}\times 100 = 1$
$\dfrac{\delta g}{g}\times 100 = 2.5$

Maximum error $= \dfrac{1}{2}(1+2.5)$
$= 1.75\%$

Q2. What error in the common logarithm of a number will be produced by an error of 1% in the number? (2017-18)

Sol. Let x as any number

and $y = \log_{10} x \quad \therefore y = \log_e x \cdot \log_{10} e$

Then $\delta y = \frac{1}{x} \log_{10} e \, (\delta x)$

$= \left(\frac{\delta x}{x} \times 100\right)\left(\frac{1}{100} \log_{10} e\right) \quad \left(\text{Given } \frac{\delta x}{x} \times 100 = 1\right)$

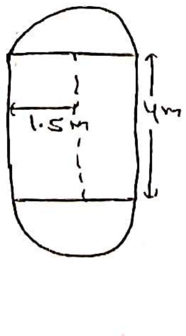
$= \frac{1}{100} (0.43429)$

$= 0.0043429$

V.Imp.

Q3. A balloon is in the form of right circular cylinder of radius 1.5 m and length 4m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon. (2013-14, 2017-18)

Sol. Here $r = 1.5$ m, $h = 4$m.,
$\delta r = 0.01$, $\delta h = 0.05$



Let V be the volume of the balloon then

$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$

$= \pi r^2 h + \frac{4}{3}\pi r^3$

Differentiating, we get

$\delta V = [\pi . 2r\,\delta r . h + \pi r^2 \delta h] + \frac{4}{3}\pi . 3r^2 \delta r$

$= \pi r [2h\,\delta r + r\,\delta h + 4r\,\delta r]$

$\frac{\delta V}{V} = \frac{\pi r [2(h+2r)\delta r + r\,\delta h]}{\pi r^2 h + \frac{4}{3}\pi r^3}$

$= \frac{2(h+2r)\delta r + r\,\delta h}{rh + \frac{4}{3}r^2}$

$= \frac{2(4+3)(0.01) + (1.5)(0.05)}{(1.5)4 + \frac{4}{3}(1.5)^2} = \frac{0.215}{9}$

$\therefore \frac{\delta V}{V} \times 100 = 2.389\,\%$

V.Imp

Q4. Find approximate value of $[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}}$ (2011-12, 2013-14)

Sol. Let $f(x,y) = (x^2 + 2y^3)^{\frac{1}{5}}$

Taking $x = 4$, $\delta x = 3.82 - 4 = -0.18$
$y = 2$, $\delta y = 2.1 - 2 = 0.1$

$\frac{\partial f}{\partial x} = \frac{1}{5}(x^2+2y^3)^{-\frac{4}{5}} . (2x) = \frac{2}{5}(4)(16+2(2)^3)^{-\frac{4}{5}} = \frac{8}{5}(32)^{-\frac{4}{5}}$

$= \frac{8}{5}(2^5)^{-\frac{4}{5}} = \frac{8}{5} 2^{-4} = \frac{8}{5} . \frac{1}{2^4} = \frac{8}{5} . \frac{1}{16} = \frac{1}{10}$

$\frac{\partial f}{\partial y} = \frac{1}{5}(x^2+2y^3)^{-\frac{4}{5}}(6y^2) = \frac{6}{5}(4)[16+2(2)^3]^{-\frac{4}{5}} = \frac{24}{5}\left(\frac{1}{16}\right) = \frac{3}{10}$

By total differentiation, we get

$$df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$= \frac{1}{10}(-0.18) + \frac{3}{10}(0.1) = -0.018 + 0.03 = 0.012$$

Hence

$$[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}} = f(4,2) + df$$

$$= [(4)^2 + 2(2)^3]^{\frac{1}{5}} + 0.012$$

$$= 2 + 0.012 = 2.012$$

Imp

Q5. Find approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2}$

Sol. Let $f(x,y,z) = (x^2+y^2+z^2)^{1/2}$

taking $x = 1$, $y = 2$, $z = 2$ so that

$\delta x = -0.02$, $\delta y = 0.01$, $\delta z = -0.06$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x = \frac{x}{(x^2+y^2+z^2)^{1/2}}$$

Similarly $\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{1/2}}$, $\frac{\partial f}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}}$

Now $df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}}(x\,\delta x + y\,\delta y + z\,\delta z)$$

$$= \frac{1}{(1+4+4)^{1/2}}(-0.02 + 0.02 - 0.12)$$

$$= \frac{1}{3}(-0.12) = -0.04$$

$$[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2} = f(1,2,2) + df$$

$$= (1+4+4)^{1/2} - 0.04 = 3 - 0.04 = 2.96$$

Some Practice Questions.

Q6. The formula, $V = kr^4$, says that the volume V of the fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V? (2012-13)

Ans - 40% increase.

Q7. If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%. (2015-16)

Q8. Find the percentage error in measuring the volume of a rectangular box when the error of 1% is made in measuring the each side. (2016-17)

Ans 3%

Q9. A balloon in the form of right circular cylinder of radius 1.5 m and length 4 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m find the % change in the volume of balloon. (2017-18)

Ans 1.55%

Q10 If $RI = E$ and possible error in E and I are 20% and 10% respectively, then find the error in R. (2018-19)

Ans 10%

5 Year's

University Paper Questions

(AKTU Question Bank)

5 Years AKTU University Examination Questions		Unit-3	
S. No	Questions	Session	Lecture No
1.	Expand $f(x,y) = y^x$ about $(1,1)$ upto second degree terms and hence evaluate $(1.02)^{1.03}$. (Long question)	2012-13	17-22
2.	Expand $e^x \log(1+y)$ in the powers of x and y upto terms of third degree. (Long question)	2014-15	17-22
3.	Express the function $f(xy) = x^2 + 3y^2 - 9x - 9y + 26$ as Taylor's Series expansion about the point (1,2). (Long question)	2017-18, 2016-17	17-22
4.	State the Taylor's Theorem for two variables. (Very Short)	2018-19	17-22
5.	Locate the stationary point of: $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. (Short question)	2012-13	17-22
6.	Find the stationary point of: $f(x,y) = 5x^2 + 10y^2 + 12xy - 4x - 6y + 1$. (Very Short question)	2013-14	17-22
7.	Find the maximum value of the function $f(xyz) = (z - 2x^2 - 2y^2)$ w□ere $3xy - z + 7 = 0$. (Very Short question)	2016-17	17-22
8.	Find the stationary point of $f(x,y) = x^3 + y^3 + 3axy, a > 0$. (Very Short)	2018-19	17-22
9.	Using the Lagrange's method, find the maximum and Minimum distances from the origin to the curve $3x^2 + 4xy + 6y^2 = 140$. (Short question)	2011-12	17-22
10.	Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (Long question)	2013-14	17-22
11.	Find the maximum and minimum distance of the point (1, 2, -1) from the sphere $x^2 + y^2 + z^2 = 24$. (Long question)	2018-19, 2017-18	17-22
12.	Find the volume contained in the solid region in the first Octant of the ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (Long question)	2013-14	17-22
13.	A rectangular box open at the top is to have 32 cubic ft. Find the dimensions of the box requiring least material for its construction. (Long question)	2014-15	17-22
14.	Using the Lagrange's method to find the dimension of rectangular box of maximum capacity whose surface area is given when (a) Box is open at the top (b) Box is closed. (Long question)	2015-16	17-22
15.	Using Lagrange's method of Maxima and Minima, find the shortest distance from the point (1, 2, -1) to sphere $x^2 + y^2 + z^2 = 24$. (Long question)	2017-18	17-22
16.	If $x + y + z = u, y + z = uv, z = uvw$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. (Long question)	2015-16	17-22
17.	If $u_1 = \frac{x_2 x_3}{x_1}, u_2 = \frac{x_1 x_3}{x_2}, u_3 = \frac{x_2 x_1}{x_3}$, find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$. (Short question)	2017-18	17-22

18.	If $u = x(1-y), v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (Very Short)	2019-20	17-22
19.	If $x = e^v \sec u$, $y = e^v \tan u$, then evaluate $\frac{\partial(x,y)}{\partial(u,v)}$. (Very Short)	2020-21	17-22
20.	Calculate $\frac{\partial(u,v)}{\partial(x,y)}$ for $x = e^u \cos v$ and $y = e^u \sin v$. (Very Short question)	2011-12	17-22
21.	If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then show that $JJ' = 1$. (Short question)	2013-14	17-22
22.	Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$. (Long question)	2014-15	17-22
23.	If $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$ (Long question)	2016-17	17-22
24.	If $= r\cos\theta, y = r\sin\theta, z = z$ then find $\frac{\partial(r,\theta,z)}{\partial(x,y,z)}$. (Very Short)	2018-19	17-22
25.	Are the functions: $u = \frac{x-y}{x+z}, v = \frac{x+z}{y+z}$ functionally dependent? If so, find the relation between them. (Short question)	2011-12	17-22
26.	If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^2 + (\lambda - z)^2 = 0$, In λ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. (Long question)	2015-16	17-22
27.	Find the relation between u, v, w for the values $u = x + 2y + z; v = x - 2y + 3z; w = 2xy - zx + 4yz - 2z^2$. (Short question)	2016-17	17-22
28.	If u, v, w are the roots of the equation $(x-a)^3 + (x-b)^2 + (x-c)^2 = 0$, then find $\frac{\partial(u,v,w)}{\partial(a,b,c)}$. (Long question)	2018-19	17-22
29.	If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ and $u + v + w = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$. (Long question)	2019-20	17-22
30.	If $u^3 + v + w = x + y^2 + z^2$, $u + v^3 + w = x^2 + y + z^2$, $u + v + w^3 = x^2 + y^2 + z$, Show that: $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1 - 4xy(xy+yz+zx) + 16xyz}{2 - 3(u^2+v^2+w^2) + 27u^2v^2w^2}$. (Long question)	2020-21	17-22
31.	Find approximate value of: $[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}}$. (Short question)	2011-12, 2013-14	17-22
32.	The formula, $V = kr^4$, says that the volume V of the fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V? (Very Short question)	2012-13	17-22
33.	If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%.	2015-16	17-22

	(Very Short question)		
34.	Find the percentage error in measuring the volume of a rectangular box when the error of 1% is made in measuring the each side. (Long question)	2016-17	17-22
35.	A balloon in the form of right circular of radius 1.5m and length 4m is surmounted by hemispherical ends. If the radius is increased by 0.01m find the percentage change in the volume of the balloon. (Long question)	2017-18	17-22
36.	What error in the logarithm of a number will be produced by an error of 1% in the number? (Very Short)	2017-18	17-22
37.	If $RI = E$ and possible error in E and I are 20% and 10% respectively, then find the error in R. (Very Short)	2018-19	17-22